Pattern MIS-recognition

By Jerome Dancis

Recognizing and extending "problem patterns" permeates the learning and doing of mathematics. Students need to be able to "recognize familiar underlying patterns within a problem". [1] This will enable them to "[apply] strategies that have worked well for them in the past" and to "generalize from strategies they have used successfully". Recognizing "problem patterns" is observing that a significant part of the problem being solved is quite similar to a significant part of a previously solved problem. [2]

In contrast, the current ubiquitous "pattern recognition [of simple numerical patterns]" is a very minor part of recognizing "problem patterns". I recognized and extended "problem patterns" daily when studying for both my Math and Physics courses during my undergraduate college years; I was majoring in both Math and Physics. I rarely (if ever) used "pattern recognition" in college.

School "pattern recognition" problems may be divided into two groups, those which are mathematically correct and those which are mathematically *wrong* ; the latter are harmful. Solutions to mathematically-correct pattern recognition problems may be divided into two groups, namely, mathematically correct solutions and mathematically *wrong* solutions; the latter are harmful. Sometimes, even mathematically-correct solutions, to mathematically-correct pattern recognition problem, are not a good use of student time and sometimes they are misleading. Examples of all these types occur in this report. Unfortunately, for our children, the harmful stuff is popular, the non-harmful stuff is not.

The National Council of Teachers of Mathematics (NCTM) latest Standards (2000) includes mathematically *wrong* solutions, for example, Incorrect Solution #3 for Problem 5, herein. Also NCTM's latest Standards discussion of "pattern recognition" contains much that is *counter-productively inadequate*. State standards and the National Assessment of Educational Progress (NAEP) math exams are largely modelled on NCTM's Standards of 1989 and 2000. NAEP's exams contain mathematically *wrong* "pattern recognition" problems, for example, Problems 14 and 16, herein. Some *not useful* pattern recognition problems are Problems 12, 19, 22 and 24. Some pattern recognition problems with unexpected solutions are Problems 8, 10 and 11. A problem, where using pattern recognition is the long way to do it, is Problem 6.

[1] Quotes in this paragraph are from Dr. Mel Levine's book, "Educational Care: A System for Understanding and Helping Children with Learning Differences at Home and in School" (Page 174) Dr. Mel Levine is Professor of Pediatrics and director of the Clinical Center for the Study of [Child] Development at the University of North Carolina.

[2] For example, students should recognize that the same problem type or pattern is shared by Problems 16-19 in my article: "Supposedly Difficult Arithmetic Word Problems", on my website at http://www.math.umd.edu/ jnd/Difficult_Word_Problems.html. The solutions, of most of the problems in that article are obtained by generalizing and extending strategies used successfully in preceding problems therein.

People commonly generalize after observing a pattern in a few cases, like all owners, of BMW cars, drive too fast. This is especially true for children. Small children will use the word "cow" to describe horses as well as cows, due to the similarity of the pattern of their shapes and coloring. Many students believe the incorrect, so-called "Strong Law of small numbers", something like, if some statement is correct a few times, and no incorrect situation has been observed, then it is always correct. All this is why, it is crucial for mathematics instruction to challenge, *not encourage* this natural tendency to generalize too easily, often with certainty, without checking.

Finding patterns should start with the patterns of English grammar. These patterns are the only patterns of crucial importance to children in Grade 1 and to English learners. Students should also learn the counterexamples. The word is "better", *not* "gooder", no matter that "gooder" fits the pattern for almost all of the first thousand comparitive words that a child learns.

Mathematics is crucial in science and the social sciences because it allows one to capture precisely the relationship between different quantities. For this reason, formulating formulas which descibe situations is a basic and significant part of mathematics. The *analysis* of patterns is a tool for finding formulas. Mathematicians and scientists do pattern *analysis*, which involves *analyzing* the situation or the scientific context, *not* just pattern *recognition.*

Finding patterns, in math, should start with realizing and understanding that

$$3 + 5 = 8 \ \textit{leads to} \ 3 + 15 = 18 \ \textit{and} \ 3 + 45 = 48, \ \textit{etc.}$$

This arithmetic pattern has *significant bonuses* as it leads to shortcuts in calculation and then contributes to the standard algorithm for adding multi-digit numbers. This arithmetic pattern also leads to counting by twos and by fives.

Students, who can count to twenty, might discover the pattern by doing a modest number of problems similar to the next problem:

Problem 1 *Add these pairs of numbers.*

$$\begin{array}{lcl} 2+3 & \textit{and} & 2+13 \\ 2+4 & \textit{and} & 2+14 \\ 4+5 & \textit{and} & 4+15 \end{array}$$

Do you notice anything of interest?

Important patterns arise in averaging. That the

$$\textit{Average}\{2,4,9\} \ = \ 5 \ \textit{leads to Average}\{982,984,989\} \ = \ 985$$

and it leads to Average {2000,4000,9000} = 5000.

Again, this arithmetic pattern has *significant bonuses* as it leads to shortcuts in calculation and more important, it leads to intuition, alias number sense, about averages. Later, this should be stated as a formula in symbolic notation, and then the formula should be proven.

Formula.

$$Average\{k + ca, k + cb, k + cd\} = k + c \times Average\{a, b, d\}$$

Students can be led to discovering these patterns for averaging, by assigning a modest number of averaging problems like:

Problem 2 *Average these sets of numbers.*

$$\begin{array}{ccccc} 2,4,9 & and & 12,14,19 & and & 32,34,39 \\ 2,4,9 & and & 20,40,90 & and & 200,400,900 \end{array}$$

Do you notice anything of interest?

When students start doing arithmetic with negative numbers, they may apply this pattern recognition on averages, to the next problem.

Problem 3 *Average these test scores:* $75, 79, 83, 87$.

Solution. Rewrite these numbers as $(80-5), (80-1), (80+3), (80+7)$, then

$$Average\{75,79,83,87\} = 80 + Average\{-5,-1,3,7\ \} = 80 + \frac{-6+10}{4} = 80 + 1 = 81.$$

Three very common types of sequences are (a) those for which *each* term is a *fixed* number more (or less) than the preceding term and (b) those for which *each* term is a *fixed* multiple of the preceding term and (c) sequences with the form, cn^2, when n is a positive integer. Type (a) are called arithmetic progressions and Type (b) are called geometric progressions. It is important that students do *not* end up thinking that these are the *only* type of sequences.

An example follows:

Problem 4 *The Fair Meadows County Library charges a fine of a fixed amount per day for each day a book is overdue. If the fine is $0.30 for a book that is two days late and $0.45 if it is three days late. What is the late fee for a book that is returned one day late? Four weeks late?*

Remark. Including the phrase "for each day" is *crucial.* My local library charges a fine of 15 cents a day *except* for the 100^{th} day, which has an extra fee of $10 to cover the cost of sending the account to a collection agency.

The next problem is largely copied from the NCTM's Principles and Standards for School Mathematics [PSSM] (2000) standard on patterns for grades 3-5 (Page 160), wherein it is stated twice, once for the particular case of "fifty" cubes and once for the general case.

Problem 5 *"What is the surface area of a tower with [fifty] [unit] cubes (including the bottom)?" Each tower has a single cube for each story (alias level).*

Correct Solution # 1. The PSSM 2000 standards state: "Fifth graders should be challenged to justify a general rule [for the geometric situation] with reference to the geometric model, for example, 'The surface area is always four times the number of [unit] cubes and one extra at each end of the tower.' " This can then be rewritten as the formula: Let $A(n)$ be the surface area for a tower of n unit cubes. $\{\text{Surface area for tower with height } n\} = A(n) = 4n + 1 + 1$.

Remark. This PSSM solution is correct only for *unit* cubes, but the word "unit" is missing from the PSSM statement of the problem.

Correct Solution # 2. One observes when adding one additional [unit] cube onto the top of the tower, that the top face of the tower and the bottom face of the new cube cover each other and that there are five new uncovered faces on the top cube. Let $A(n)$ be the surface area for a tower of n unit cubes. Thus

$$A(n) \;=\; A(n-1) - 1 + 5 \;=\; A(n-1) + 4.$$

Thus the sequence of surface areas is an arithmetic progression. Hence the formula for arithmetic progression apply. Then, our knowledge of arithmetic progressions provides: $A(n) = 6+4(n-1)$.

Remark. This solution included a recursive observation. This is useful background for recursion in computer science, and for proof by mathematical induction.

Incorrect Solution # 3. The NCTM 2000 standards state "Fourth graders might make a table

Number of cubes (n)	Surface area in square units (S)
1	6
2	10
3	14
4	18

and note the iterative nature of the pattern. That is, there is a consistent relationship between the surface area of one tower and the next bigger tower: 'You add four to the previous number'."

Warning. Students should *not* be trained to make the *unwarrented* assumption that "a consistent relationship between the surface area of one tower and the next bigger tower" for just four towers implies the same consistent relationship between the surface area of one tower and the next bigger tower, for *all* towers.

It is important in discussing and in testing patterns that the basic idea of mathematics as a rigorous subject involving disciplined reasoning not be lost.

The next example is largely copied from the NCTM's Principles and Standards for School Mathematics [PSSM] (2000) standard on patterns for grades 3-5 (Pages 159 and 160)

Example 6 *Suppose students are to investigate the sum of the first n odd numbers*

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$

and notice that the answer is the perfect square n^2.

Remark. It is important that students learn that this is a mathematical fact which is *not* established for all n simply by checking some cases,and that it is *not even* established for $n = 5$, by checking the cases $n = 1, 2, 3, 4$.

PSSM's concluding paragraph on this example is: "Examples like this one give the teacher important opportunities to engage students in thinking about how to articulate and express a generalization – 'How can we talk about how this pattern works for a square of any size?' Students in Grade 3 should be able to predict the next element in a sequence by examining a specific set of examples. By the end of fifth grade, students should be able to make generalizations by reasoning about the structure of the pattern. For example, a fifth-grade student might explain that 'if you add the first n odd numbers, the sum is the same as $n \times n$.' "

Of course, all this is useful, but by itself, it is *counterproductively inadequate.*

Omitted is:

* The teacher engaging students in thinking about how to verify the *correctness* of the articulated generalization.

* Students (or the teacher) *verifying* the generalizations they made by reasoning about the structure of the pattern.

* A fifth-grade student (or the teacher) *justifying* the formula that 'if you add the first n odd numbers, the sum is the same as $n \times n$.' " The *difference* between noticing a pattern and concluding that the observed pattern continues is crucial, but this is *overlooked* in PSSM.

Thus PSSM's concluding paragraph, on this problem, is advocating the training of students to generalize but *not* to check and verify. This is a useful problem, with popular harmful solutions.

Remark. The PSSM presentation omits the following crucial observation.

Observation. The n-th odd number is $2n - 1$.

One-line Proof: that the sum, of the first N odd numbers, is $N \times N$.

The odd numbers form an arithmetic progression. Hence, the sum, of the first N odd numbers, is $\frac{N\times(1+[2N-1])}{2} = N^2$.

Remark. The pictorial proof presented in PSSM (Page 159) is a "proof by accident", not something to count on.

Remark. That the sum, of the first n odd numbers, is $n \times n$ is a mathematical *curiosity*. I first learned it when reading the PSSM book. Learning curiosities should be *low* on the priority list.

Doubling sequences. That a sequence, which starts with say $1, 2, 4$ may lead to unexpected numbers is demonstrated by the next two problems.

Problem 7 (Incorrect) *A sequence, begins with* $1, 2, 4$*; if the pattern continues, what is the next number?*

Remark. Many math teachers are aware that many different sequences begin with $1, 2, 4$; but also believe that when the phrase "if the pattern continues" is included, then this problem has a *unique* answer. BOO!

The point here is that though context may allow one to select a particular next term which is most likely to fit a given collection of data or to apply to a given specific situation, it is simply *incorrect* to say that there is a single next answer to the problem in the absence of additional information. The phrase "if the pattern continues", by itself, does *not* specify *which* one, of many possible patterns, is the one being continued.

Example 8 *A pattern that begins with* $1, 2, 4,$ *is the one supplied by the digits of the decimal expansion of* $\frac{124}{999}$, *it goes* $1, 2, 4, 1, 2, 4, 1, 2, 4, \cdots$.

Problem 9 (Correct) *A sequence, begins with* $1, 2, 4$*; the sequence is a geometric progression [or an arithmetic progression or of type* cn^2, $\forall_{c \in Z}, \forall_{n \in R^+}$*], what is the next number?*

Remark. The conditions, that the sequence is a geometric progression [or an arithmetic progression or of type cn^2], *restricts* the sequence to being the doubling sequence. No other sequence, among these three types, starts with $1, 2, 4$.

Problem 10 *Let us count the number of regions that lines in "general position" divide the plane into. For instance, if there were no lines, there would be 1 region—namely, the entire plane; if there were one line, the two sides of that line would make 2 regions; two intersecting lines would divide the plane into 4 regions (here "general position" means not parallel). The connection between small numbers of lines and the numbers of connected regions is given in this table:*

Number of lines	*Number of regions*
0	1
1	2
2	4

How many regions will result when 3 lines divide the plane?

Answer. The answer happens to be 7, *not* 8!

Problem 11 *Consider a circle, with several points selected on its circumference. All the chords, whose endpoints are the selected points have been drawn (and no other). These chords will divide the circle into a number of connected regions. The connection between small numbers of points on the circumference and the numbers of connected regions is given in this table:*

Number of points	*Number of regions*
1	1
2	2
3	4
4	8
5	16

How many regions will result when 6 points are selected on the circumference?

Answer. The answer happens to be 31, *not* 32!

Rabbits. The next problem is simple to state; it is connected to a story about rabbits having so many baby rabbits, that the n^{th} number, in the sequence, will the rabbit population be in year $\#n$, starting with the number 2.

Problem 12 *The Fibonacci sequence is* $1, 1, 2, 3, 5, 8, \cdots$*, where each term (after the second) is the sum of the preceding two terms; also the first two numbers are both one. Fibonacci found the formula for the* 100^{th} *term 8 centuries ago. Now it is your turn.*

Remark. In spite of the simplicity of the statement, students trying pattern recognition on this problem, will mostly experience frustration.

Problem 13 *The table below shows the change in population for a group of rabbits over a 4-year period.*

Population of rabbits *(at end of year)*

year	1	2	3	4	5
Population	2	4	8	16	32

If this pattern continues, what will be the population of rabbits in year 7?

Again, the phrase, "If this pattern continues" does not specify which pattern is the one being continued.

The simplest arithmetic pattern (but not the simplest biological pattern) is that the number of rabbits doubles each year. But the biology is skipped, namely, the assumption that the growth rate is proportional to the population.

Possible Biology Pattern. Assume the following: Exactly half the population is female and each female gives birth to one female and one male rabbit each year. The original 2 rabbits were born in year #1 and their parents died that year. Also, each rabbit dies at age 6 years. Then the population for the first 5 years will be the same as in the table above, but as the older rabbits die; the population will *less than* double each year, thereafter.

I cannot think of a Possible Biology Pattern, which (i) is half as simple as this one, and (ii) which satisfies the given table, and (iii) which has the population contiuing to double each and every year. Rabbits living happily ever after and never dying *not* permitted.

Problem 14 (1992 NAEP mathematics assessment) [3]

[3] The National Research Council published two books on NAEP. The second one, "Grading the Nation's Report Card - Research from the Evaluation of NAEP" contains the papers solicited for the committee which was writing

Puppy's Age	*Puppy's Weight*
1 month	10 *lbs.*
2 months	15 *lbs.*
3 months	19 *lbs.*
4 months	22 *lbs.*
5 months	? *lbs.*

John records the weight of his puppy every month in a chart like the one shown above. If the pattern of the puppy's weight continues, how much will the puppy weigh at 5 months?

Expected solution. The trick is to observe that the monthly weight *increases* are $5, 4, 3,$ which is counting backwards. Make the *unwarrented* assumption, that this counting backwards continues will result in the puppy gaining 2 lbs. in the next month and therby weighing $22+2 = 24$ lbs. at 5 months. [4]

A formula, which will produce the table given in the problem, but provide a different weight in Month 5, is:

Unexpected solution. Suppose that the pattern of the puppy's weight is given by

$$f(n) = \frac{1}{2}[-n^2 + 13n + 8 + (n-1)(n-2)(n-3)(n-4)],$$

then the puppy's weight at 5 months will be 36 lbs. Quite a growth spurt. A more modest growth spurt is provided by the formula: $f(n) = -\frac{1}{2}n^2 + \frac{13}{2}n + 4 + \frac{1}{6}(n-1)(n-2)(n-3)(n-4)$, which results in the puppy's weight becoming 28 lbs in Month 5.

The absurdity of expecting the counting backwards pattern, for weight increases, to continue, is demonstrated by projecting further into the future:

Problem 15 *In the preceding problem, If this counting backwards pattern for weight increases continues, how much will the puppy weigh at 14 months?*

Answer. -3 *lbs*, since the weight increases will be $5, 4, 3, 2, 1, 0, -1, -2, -3, -4, -5, -6, -7$.

NAEP sample problem are available at http://nces.ed.gov/nationsreportcard/ITMRLS/pickone.asp

Problem 16 (NAEP Grade 8 sample problem (2003)) *#41:*

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the report. There is one article on mathematics, "Families of Items in the NAEP Mathematics Assessment" by Patricia Ann Kenney, in which she suggests a way to make this problem easier. The observation that this problem is incorrect was not made.

[4] That less than 30% of students obtained the correct answer probably means that their teachers had not provided them with the rote scheme for looking at these types of progressions.

Term	1	2	3	4
Fraction	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$

If this list of fractions continues in the same pattern, which term will be equal to 0.95*?*

Multiple choice: A) The 100^{th}*, B) The* 95^{th}*, C) The* 20^{th}*, D) The* 19^{th}*, E) The* 15^{th}

Expected solution. Observe that *a simple* pattern has n^{th}-term: $\frac{n}{n+1}$, and that $0.95 = \frac{19}{20}$. Hence, the answer is: D) The 19^{th}-term.

Only about one in four (27%) of the Grade 8 students, obtained the correct answer. Absurd.

Again, this is a mathematically incorrect problem as the pattern that is continuing was not specified.

Unexpected, but also correct solution. E) The 15^{th}-term, when the formula for the n^{th}-term of the pattern is:

$$\frac{80 \times 14 \times 13 \times 12 \times 11n + (n+1)(n-1)(n-2)(n-3)(n-4)}{80 \times 14 \times 13 \times 12 \times 11(n+1)}$$

Remark. There is a pattern to this formula, which makes it easy to modify in order to make choices A, B or C, additional correct answers.

Problem 17 *[A NAEP Grade 8 Problem]* [5] *A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. The pattern continues infinitely.*

Marcy has to determine the number of dots in the 20th step, but she does not want to draw all 20 pictures and then count the dots. Explain or show how she could do this and give the answer that Marcy should get for the number of dots. [6]

[5]NAEP Grade 8 Problem 139. NAEP questions may be accessed from http://nces.ed.gov/nationsreportcard/ITMRLS/pickone.asp NAEP is "the nation's report card".

[6]The sentence, "Assume that the number of dots added at each step is more than the number added in the previous step" does rule out many otherwise possible correct answers, but still the number of dots in the 20^{th} term may be 267 *or any larger* positive number. This and more on this problem on Page 1 of James Milgram's article on patterns, on his website, http://math.stanford.edu/ftp/milgram/pattern-problems.pdf. Also read Ralph Raimi's comment on Page 11 therein.

This problem is quite similar to one listed by the committee writing the parameters for President Clinton's proposed national mathematics test in mathematics. The committee's incorrect solution was something like this:

Incorrect Method of Solution # 1. Count the number of dots in each set; make a table:

Set #	1	2	3
Number of dots	2	6	12

Step 0. Implicitly, make the *unwarrented* assumtion that the pattern is described by a polynomial formula of degree 2.

Step 1. Use the method of successive partial differences. This will provide the formula $f(n) = n^2 + n$. Hence the 20^{th} term will have 420 dots.

Remark. Of course, most students will use the method of successive partial differences as a *rote procedure. No* understanding necessary.

Remark. Making the unwarrented assumtion that a pattern is described by a polynomial formula is *wrong* mathematics, even if one often *lucks out* and gets the right answer. Worse it is *wrong* science when the pattern comes from science.

Remark. Fortunately, it is easy to write a mathematically correct version of this problem, namely:

Problem 18 (correct) *A pattern of dots is shown below. At each step, more dots are added to the pattern. The number of dots added at each step is more than the number added in the previous step. You will extended the pattern infinitely.*

Describe a possible extension of this pattern. Then draw the 20^{th}*- picture, without drawing any of the* 4^{th} *through* 19^{th}*-pictures.*

Remark. Note that here, "a possible extension of this pattern" is requested, not *the non-existent* unique extension.

Remark. Also here a description of the 20^{th}- picture is required. *By itself*, the formula, $f(n) = n^2 + n$, of Solution #1 provides *no* desciption of the geometry of the n^{th} set of dots.

A correct solution to Problem 18. This is also almost a reasonable Method of Solution to Problem 17. Count the number of rows and the number of columns in each set; make a table:

Set #	1	2	3
Number of rows	1	2	3
Number of columns	2	3	4

It "appears" that

{ the number of rows} = {the set #}; and { the number of columns} = 1 + {the set #}.

We observe that each set, of dots, forms a rectangular lattice, hence

{ the number of dots} = { the number of rows} × { the number of columns}

or in function notation $f(n) = n(1+n)$.

Remark. Note that this solution requires a geometric analysis, albeit a simple one, but a geometric analysis, none the less. Training in geometric analysis is important. In contrast, Solution # 1 requires *no* geometric analysis.

Remark. Of course, if these sets of dots came from a problem in science, one needs to use the science to *justify* that the scientific situation forces

{ the number of rows} = {the set #}; and { the number of columns} = 1 + {the set #}.

Rashly assuming either equation would be *wrong science.*

Remark. The three sets shown are all *single level* rectangular sets. It is possible that the pattern of the number of levels is $1, 1, 1, 7, \cdots$; this would occur when the number of levels is determined by the formula, $f(n) = 1 + (n-1)(n-2)(n-3)$. This would result in the 20^{th} set having $20 \times 21 \times (1 + 19 \times 18 \times 17)$ points.

Of course, all sorts of other things might occur, like a hole appearing in a set of dots or a half dot appearing.

Problem 19 (Not useful) *Find 5 patterns among the entries of Pascal's Triangle:*

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
\vdots
\end{array}
$$

Remark. This problem makes no hidden assumptions. The issue is that there is no useful gain in mathematical knowledge for students who may notice that entries of the rows add up to powers of 2, etc. Prior to work done counting combinations, such investigations do little to contribute either to the mastery of core topics or to the development of mathematical reasoning skills, and as such is of limited value. Pattern hunting in Pascal's triangle, will *not* lead to the important formula for the entries in Pascal's Triangle. *Worse*, this problem is training students to look for patterns frivilously, usually outside of any useful context.

This problem is pretentious busywork, with likely, little success for many students. It is poor use of students time.

Remark. It is important that math instruction help and guide students to obtaining good taste in when it might be useful to look for a pattern and when not. The last problem deserves a very low taste rating.

Half-price Sales.

Problem 20 *The store is having a "Buy one get a second at half price sale", when both items have the same price. John buys two shirts, price \$10 for one and two ties, price \$5 for one. What percentage saving does John obtain on his purchase of the two shirts? on his purchase of the two ties? What pattern does John observe?*

John will observe that the saving was 25% in both cases. But, to conclude that the savings *is always* 25%, one needs to do the following algebraic problem:

Problem 21 (Algebraic version) *A store is having a "Buy one get a second at half price sale", when both items have the same price. What percentage saving is obtained on the purchase of any two identically priced items?*

In a beginning algebra class, Problem 20 might be assigned as background for Problem 21; but students should progress to the point, where they are doing Problem 21, without Problem 20 as a prompt.

While shopping, John might do the calculations of Problem 20, which hopefully would suggest to him to formulate and do the algebraic Problem 21.

Now for real-world versions of these two problems:

Problem 22 *The store is having a "Buy one get another at half price sale", when the second item has the same or lesser price. John buys two shirts, priced at \$8.99 and 10.98 and two ties, priced at \$3.95 and \$5.97. What percentage saving does John obtain on his purchase of the two shirts? on his purchase of the two ties? What pattern does John observe?*

Solutions. Here, John saves about 22.5% on the shirts and about 20% on the ties. *Not* likely that John will notice a pattern, even if he collects more data. This is not a useful pattern recognition problem. But there is a useful algebraic version, namely:

Problem 23 (Algebraic version) *The store is having a "Buy one get another at half price sale", when the second item has the same or lesser price. What can you say about the possible percentage savings?*

Using Algebra with inequalities, a student should show that the savings is at most 25%, with a savings of exactly 25% occuring when the two items have the same price and a *less* than 25% savings, when the two items have different prices.

Rules for Exponents. After having defined $x^n,\ \forall_{n\in Z^+}$ *and* $\forall_{x>0}$ [7], the question arises: What should the definitions of x^0 *and* x^{-1} be? It is tempting to do the next pattern recognition problem.

Problem 24 (Correct, but not useful mathematics) *Having defined* $x^n,\ \forall_{n\in Z^+}$ *and* $\forall_{x>0}$*, look at the sequence,* $10^5, 10^4, 10^3, 10^2, 10^1$*; spot a pattern and use it to define* 10^0 *and* 10^{-1}.

Solution. Yes, the sequence, $10^5, 10^4, 10^3, 10^2, 10^1$ is a geometric sequence, with $\frac{1}{10}$ as the constant multiple between each term and its succsessor. So *if this pattern continues*, the next two terms will be $\frac{1}{10} \times 10 = 1$ *and* $\frac{1}{10} \times 1 = \frac{1}{10}$. So *if this pattern continues*, 10^0 should be 1, and 10^{-1} should be $\frac{1}{10}$.

Remark. That the pattern does so continue is nice and it is a consequence of the fabulous consistency of mathematics. But, mathematics does *not* assume that the pattern does so continue. In mathematics, the definition is based on preserving

The Basic Rule for Exponents. $x^n\ x^m = x^{n+m},\ (\forall_{x>0})$,

Problem 25 (Useful mathematics) *Having defined* $x^n,\ \forall_{n\in Z^+}$ *and* $\forall_{x>0}$*, and having established The Basic Rule for Exponents, for positive exponents, determine what definitions of* x^0 *and* x^{-1} *will be consistent with The Basic Rule for Exponents.*

Correct Solution. Ask, what is zero? Answer, it is the additive identity, that is $n + 0 = n,\ \forall_{n\in Z}$. Use this with The Basic Rule for Exponents:

$$x^n = x^{n+0} = x^n\ x^0 \implies x^0 = 1,\ \forall_{x>0}.$$

[7] The math symbol, "$\forall$" denotes the phrase "for all" or "for each".

Ask, what is -1? Answer, it is the additive inverse of 1, that is $-1+1=0$. Use this with The Basic Rule for Exponents and the just defined $x^0=1$:

$$1 = x^0 = x^{-1+1} = x^{-1}\ x^1 \implies x^{-1} = \frac{1}{x},\ \forall_{x>0}.$$

In the same way, one obtains that $x^{-n} = \frac{1}{x^n},\ \forall_{x>0}$ *and* $\forall_{n\in Z^+}$.

Remark. Having obtained these definitions for $x^n,\ \forall_{n\in Z}$, one still needs to check that The Basic Rule for Exponents is valid $\forall_{n\ and\ m\in Z}$.

All this leads us to a crucial principle:

Test and homework problems involving patterns and sequences should not rely on unstated assumptions, nor should they imply that there is a unique answer when this is not mathematically justified. In particular, a given sequence of numbers with no other information stated has infinitely many continuations.

Adhering to this prinicple is crucial if students are to be fairly tested, are to develop proper notions of mathematical reasoning, and if they are to find mathematics presented as a series of logical steps rather than rules without justification.

In fact, claiming that an observed pattern generalizes without justification not only misleads students about the role of mathematical reasoning in understanding and using mathematics, it can also lead to false conclusions.

These incorrect problems and solutions are harmful since they in fact deliver a false impression of mathematics. Worse, they deliver a false impression of science and deliver *wrong* science as students spot incorrect patterns, which are valid for a few cases.

High school students should do the following two problems as a partial cure for the habit that all progessions are arithmetic or geometric progessions or of type cn^2. A friend was trained in pattern recognition in middle school. So she, dutifully, looked for and spotted some original patterns while taking calculus in college. After seeing many of her answers, based on pattern recognition marked wrong, she stopped looking for patterns and did well in calculus. Other students continue to spot *incorrect* patterns, to their detriment.

Problem 26 *Find a formula, which describes a sequence, which begins with* $1, 2, 4$, *but the next number is* $-\sqrt{2}$.

Answer. An answer is $2^{n-1} + (-\sqrt{2} - 2^{4-1})\frac{(n-1)(n-2)(n-3)}{(4-1)(4-2)(4-3)}$.

Problem 27 *Find a polynomial formula, which describes a sequence, which begins with* $1, 2, 3, 4$, *but the next number is* $-\sqrt{2}$.

Answer. An answer is $n + (-\sqrt{2} - 5)\frac{(n-1)(n-2)(n-3)(n-4)}{(5-1)(5-2)(5-3)(5-4)}$.

Bad Habits are Hard to Break.

Problem 28 (A Ponzi Pyramid Scheme) *You receive a letter from a trusted friend, with a list of ten names (with addresses) and with these directions: (*) Send one dollar to the name at the top of the list. (*) Cross off the top name on the list and place your name and address on the bottom. (*) mail the modified list with the directions to ten of your friends. How much money should you expect will be mailed to you, assuming everyone who receives this letter complies.*

Solution. $\$10^9$ = one billion dollars! Did I mention that Pyramid Schemes are illegal?

Problem 29 *The price of some stock was* \$40 *in January,* \$50 *in February,* \$60 *in March,* \$70 *in April and* \$80 *in May. What will the price be in December?*

Remark. A high school graduate, well trained in pattern recognition, but not trained to check that the perceived pattern actually continues, may conclude that the stock will be \$150 in December.

Several years ago, Math Educator, Guershon Harel noted: "During the last two years, I have worked intensively with junior-high and high-school inservice teachers. One of the most robust behaviors I see with these teachers is the absolute reliance on pictures and patterns. Almost in all problems I give to the teachers, their first approach is to attempt to solve them by numeric patterns or by drawing pictures. Once the teachers observe a 'pattern' (from two or three cases) or draw a picture, they derive the final conclusion empirically or perceptually. There are almost no attempts to support and supplement their empirical and perceptual reasoning with algebra".

Training students on pattern recognition, but not training them to check that the perceived pattern actually continues, is training them to make rash, often incorrect projections about the future or invent incorrect formulas/shortcuts in Calculus or science. What is really important is the exposure/training in non-rigorous arguments. They can less readily see through the faulty reasoning so often presented in the media and by politicians. [8] Also, they will have more difficulty adjusting to and understanding college courses.

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[8] This paraphrases a comment about Euclidean Geometry, made by Barry Simon, professsor of mathematics, in his February 6, 1998 Los Angeles Times article.